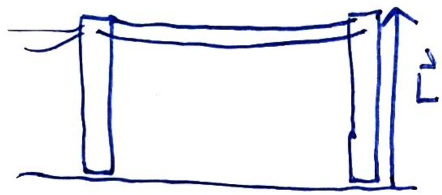
What is a vector? (Review)

Simplest definition: Something with a magnitude and a direction.

What does "something" mean? Generally, at first, you want to think of a vector as being an object that physically exists.

For example, think of a powerline pole: there is a "length vector" that extends from the bottom of the pole to the top, along the length.

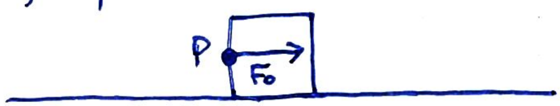


$$\vec{L} = L_0 \hat{L}$$

$\vec{L}$: length vector; L_0: length of pole; $\hat{L}$: direction

The above idea breaks down however, because vectors don't always have to correspond to an object with a physical length.

Consider pushing a box across a flat, frictionless surface, with force $\vec{F} = F_0 \hat{x}$, at point P. $\longrightarrow \hat{x}$



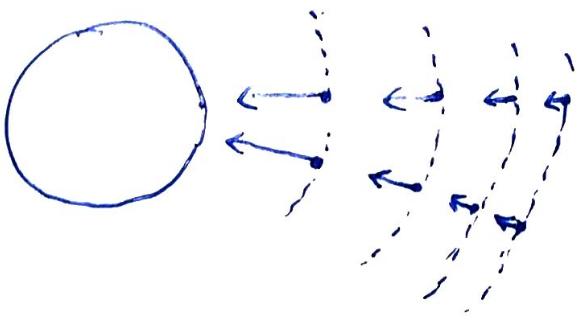
The force vector $\vec{F}$ does not extend through physical space, but really only exists at the point P.

* $\vec{F} = F_0 \hat{x}$ says that at point P, there is a force exerted in the $\hat{x}$ - direction, with magnitude F_0. *

All of the information is contained in the point P!

Consider Earth's gravitational acceleration field, $\vec{g} = \frac{GM}{r^2}\hat{r}$.
This is $\vec{F} = F_0\hat{x}$, with F_0 depending on space through r.

Now $\vec{g} = \vec{g}(r)$, and every point (r, θ, ϕ) contains information about the acceleration field.



$\vec{g}$ exists everywhere, but the vector itself does not "extend" into physical space.

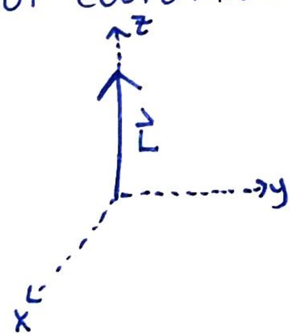
Vectors, such as $\vec{F}$ and $\vec{g}$, can still be thought of geometrically however. Their "extension" is abstract, but they define <u>geometrically rigid</u> objects (under rotation).

<u>Rotation & Vectors</u>

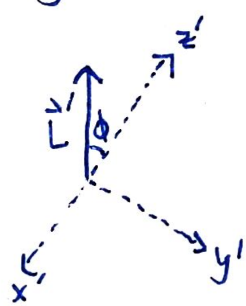
What does geometrically rigid mean?

Consider the length vector $\vec{L} = L_0\hat{L}$ of the power line pole.

Because the object is physical, it exists independent of <u>your</u> <u>choice</u> of coordinate system (only consider <u>rotation</u> for now).



Facing the pole

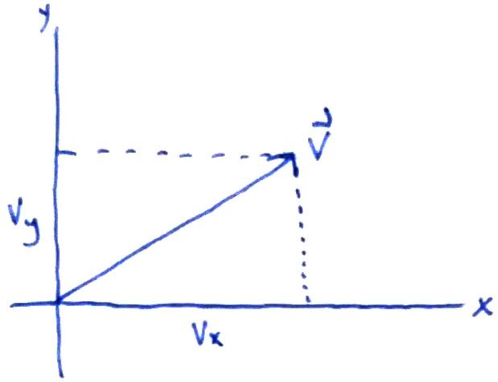


Tilting your head.

Same "quality" of the vector remains the same in the new coordinate system.

Even for abstract vectors, some "quality" remains the same (when rotating the coordinate system).

Consider a 2D orthogonal coordinate system, with an abstract vector $\vec{v}$.



Usually we write:

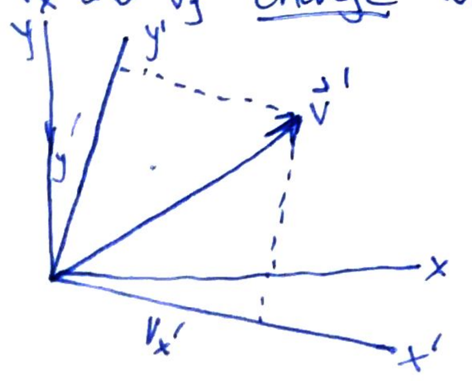
$$\vec{v} = v_x \hat{x} + v_y \hat{y}$$

where $\hat{x}, \hat{y}$ are the basis vectors in the coordinate system.

$$v_x = \vec{v} \cdot \hat{x}, \quad v_y = \vec{v} \cdot \hat{y}$$

v_x and v_y are the components and are a specific representation of the geometrical object $\vec{v}$.

v_x and v_y change when rotating (and under other transformations)



$$v_x' \neq v_x, \quad v_y' \neq v_y$$

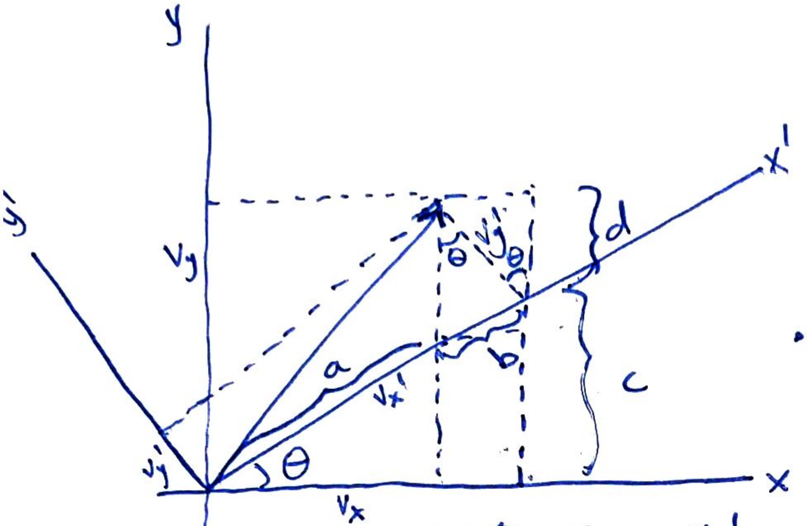
What "quality" stayed the same?

$\Rightarrow$ The norm (or magnitude).

$$|\vec{v}'| = |\vec{v}|$$

* We say: The norm of a vector is invariant under rotations of the coordinate system. *

How do we find $\vec{v}'$, and its components?



$$\cos\theta = \frac{V_x}{a} = \frac{V_y'}{d}$$

<u>x-comp</u>

$$a = V_x' - b$$

$$b = V_y' \tan\theta$$

$$\Rightarrow \quad a = V_x' - V_y' \tan\theta$$

$$\cdot\cos\theta \quad \Rightarrow \quad a\cos\theta = V_x'\cos\theta - V_y'\sin\theta$$

$$\boxed{V_x = V_x'\cos\theta - V_y'\sin\theta}$$

<u>y-comp</u>

$$C = V_x'\sin\theta \qquad d = V_y'\cos\theta$$

$$V_y = C + d = V_x'\sin\theta + V_y'\cos\theta$$

$$\boxed{V_y = V_x'\sin\theta + V_y'\cos\theta}$$

In matrix notation: $\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} V_x' \\ V_y' \end{bmatrix}$

or: $\vec{V} = R^T \vec{V}'$, can transpose to get new coordinates in terms of old.

$\vec{V}' = R\vec{V}$ } R is the rotation matrix.

$|\vec{V}|$ is <u>invariant</u> <u>under</u> <u>rotation</u>.

* In general, in any transformation that leaves the coordinate system orthogonal, $|\vec{V}|$ is invariant. *

Complex Numbers

Now that we have a good handle on vectors, we can investigate complex numbers.

$\Rightarrow$ What happens when you allow $\sqrt{-1}$ into your mathematics system?

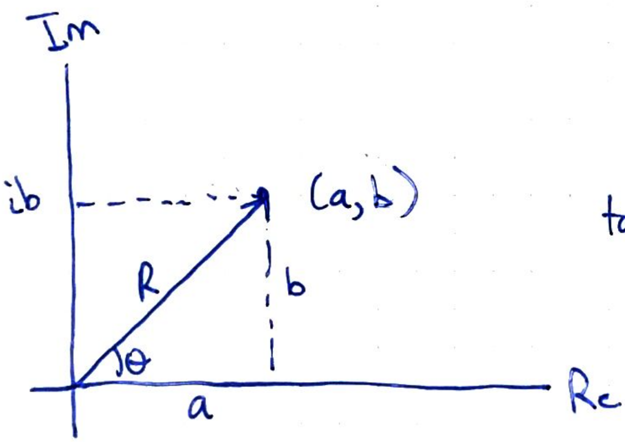
$z = 5 + 2\sqrt{-1}$ } Is there any way to add these "numbers"? $\Rightarrow$ Not really.

Treat z like a vector with components in real space and "imaginary" space. Let $i \equiv \sqrt{-1}$

$\Rightarrow \quad z = 5 + 2i$

or in general, $\quad z = a + ib \quad , \quad a, b \in \mathbb{R}$

$z \in \mathbb{C}$ (complex)



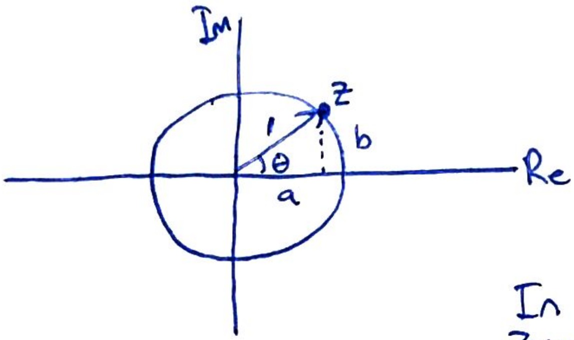
$$R^2 = a^2 + b^2 \qquad a = R\cos\theta$$
$$b = R\sin\theta$$
$$\tan\theta = \frac{b}{a}$$

So we can map $(a, b) \rightarrow (R, \theta)$

Any point $z \in \mathbb{C}$ can be described using:

$$z(R, \theta) = R\cos\theta + iR\sin\theta$$

Consider a unit circle in complex space.



$z = a + ib$

$z = (1 \cdot \cos\theta) + i(1 \cdot \sin\theta)$

$z = \cos\theta + i\sin\theta$

In this case, for a point on the unit circle, $z(\theta)$ describes rotation.

<u>Euler's Formula</u>: Euler showed $e^{i\theta} = \cos\theta + i\sin\theta$

You can "derive" this using the Taylor expansion of $e^{i\theta}$, but we really haven't defined what Taylor expansion means for complex numbers/functions.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2} + \frac{i^3\theta^3}{3!} + \ldots$$

$$= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \ldots$$

$$= \underbrace{\left(1 - \frac{\theta^2}{2} + \ldots\right)}_{\text{T.E. for } \cos\theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \ldots\right)}_{\text{T.E. for } \sin\theta}$$

$$\therefore \underbrace{e^{i\theta}}_{\text{This must encode information about rotation.}} = \cos\theta + i\sin\theta$$

This must encode information about rotation.

<u>Recall</u>: $z(a,b) = a + ib \qquad z(R,\theta) = R\cos\theta + iR\sin\theta$

$$\therefore \quad z(R,\theta) = R(\cos\theta + i\sin\theta) = Re^{i\theta}$$

$$Z(R,\theta) = R e^{i\theta}$$

R ↑ Extent from origin

$e^{i\theta}$ ← Rotation by θ

Differential Equations & Complex Numbers

You should be familiar with the following 2nd order, linear, ordinary differential equation:

$$\frac{d^2 x(t)}{dt^2} = -\omega^2 x(t) \quad \Big\} \text{ Simple harmonic motion}$$

In English: "What function $x(t)$, when you take its second derivative, gives you the function $x(t)$, times a constant?"

$\Rightarrow$ Sinusoidal functions!

$$y = \cos x \qquad \frac{d^2 y}{dx^2} = -\cos x$$

$$y = \sin x \qquad \frac{d^2 y}{dx^2} = -\sin x$$

Usually, $x(t) = A\cos(\omega t + \phi)$

↑ 2 general constants that depend on the initial conditions.

or $\quad x(t) = A\sin(\omega t + \phi)$ (cos & sin are the same function, one is offset from the other).

There is another function whose 2nd derivative gives a constant times that function: e^x

$$\Rightarrow \quad y = e^x \qquad \frac{d^2 y}{dx^2} = e^x$$

Let's try it: $x(t) = Ae^{Bt}$

$$\frac{dx(t)}{dt} = ABe^{Bt} \qquad \frac{d^2 x(t)}{dt^2} = AB^2 e^{Bt} = -\omega^2 x(t)$$

$$\Rightarrow \quad B^2 = -\omega^2$$

$$\sqrt{B^2} = \sqrt{-\omega^2}$$

$$\Rightarrow \quad B = i\omega$$

$\therefore$ one solution is $x(t) = Ae^{i(\omega t + \phi)}$ Can add this freely, doesn't affect previous argument.

We don't want imaginary numbers in a physical solution however.

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i(\omega t+\phi)} = \cos(\omega t+\phi) + i\sin(\omega t+\phi)$$

Our other solutions!

We say the solution $x(t)$ is either the Real or Imaginary part.

$$x(t) = \mathrm{Re}\{Ae^{i(\omega t+\phi)}\}$$

or $$x(t) = \mathrm{Im}\{Ae^{i(\omega t+\phi)}\}$$

Can do the math using complex exponential, take real part at the end.

Waves & Complex Numbers

From above, $x(t)$ is an oscillating function. Other oscillating functions include waves.

$$y(x,t) = A\cos(\vec{k}\cdot\vec{x} - \omega t + \phi)$$

No reason we can't represent this using complex exponentials:

$$y(x,t) = \mathrm{Re}\{Ae^{i(\vec{k}\cdot\vec{x} - \omega t + \phi)}\}$$

Math is much easier now!

Usually we have: $y(x,t) = Ae^{i(\vec{k}\cdot\vec{x} - \omega t + \phi)}$

Ket Notation

What does $|x\rangle$ mean? It is a fancy way of writing $\vec{x}$, basically.

Recall: Vector $\vec{v}$ $\qquad \vec{V} = V_x \hat{X} + V_y \hat{y}$ $\qquad V_x, V_y$ are components.

basis vectors

$$V_x = \vec{v}\cdot\hat{x} \qquad V_y = \vec{v}\cdot\hat{y}$$

$$\Rightarrow \quad \underbrace{\vec{v} = (\vec{v}\cdot\hat{x})\,\hat{x} + (\vec{v}\cdot\hat{y})\,\hat{y}}$$

I always thought this was a silly equation.

Similarly, $|x\rangle = c_1 |e_1\rangle + c_2 |e_2\rangle$ $\qquad c_1, c_2$ are components.

basis vectors

$$c_1 = \langle x|e_1\rangle \qquad c_2 = \langle x|e_2\rangle$$

$$\Rightarrow \quad |x\rangle = \langle x|e_1\rangle |e_1\rangle + \langle x|e_2\rangle |e_2\rangle$$

Now, $\langle x|e_1\rangle$ must represent the <u>dot product</u>, whatever that means at this point.

Here I introduced $\langle x|$, the Bra. For now, think of it as another vector, that you can combine with kets to make Bra-Kets.

<u>Ex</u>: $\quad |a\rangle = A|e_1\rangle \qquad \langle b| = B\langle e_1| + C\langle e_2|$

$$\langle b|a\rangle = AB \underbrace{\langle e_1|e_1\rangle}_{=1,\text{ like } \hat{x}\cdot\hat{x}=1} + AC \cancelto{0}{\langle e_2|e_1\rangle} \quad \Big\}\text{ orthogonal } \langle e_2|e_1\rangle = 0$$

$$\Rightarrow \quad \langle b|a\rangle = AB$$

There is an issue though, in $|x\rangle = c_1|e_1\rangle + c_2|e_2\rangle$,

$c_1, c_2 \in \mathbb{C}$ => complex

When you want $\langle x|$, must take complex conjugate

Ex: $|x\rangle = 5i\,|e_1\rangle$

$$\Rightarrow \quad \langle x| = -5i\,\langle e_1|$$

and
$$\langle x|x\rangle = (-5i)(5i)\,\overbrace{\langle e_1|e_1\rangle}^{1}$$
$$\langle x|x\rangle = 25$$

From this example, you should also see that similar to:

$$|\vec{v}|^2 = \vec{v}\cdot\vec{v} \qquad \big|\,|x\rangle\big|^2 = \langle x|x\rangle$$

Important Notes: 1) c_1, c_2 change when you change basis, just like v_x, v_y.

2) Because c_1 & c_2 represent probabilities, we always have to normalize our kets.

$$|x\rangle = c_1|e_1\rangle + c_2|e_2\rangle \qquad \big|\,|x\rangle\big| = \sqrt{\langle x|x\rangle}$$

$$\langle x|x\rangle = c_1^2 + c_2^2 \quad \Rightarrow \quad \big|\,|x\rangle\big| = \sqrt{c_1^2+c_2^2}$$

$$\Rightarrow \quad |x\rangle = \frac{c_1}{\sqrt{c_1^2+c_2^2}}|e_1\rangle + \frac{c_2}{\sqrt{c_1^2+c_2^2}}|e_2\rangle$$

3) When you scale $|x\rangle$ by a constant, say $a \in \mathbb{R}$, the physical state does not change!

$a|x\rangle$ represents the same state as $|x\rangle$.