Eigenvalues & Eigenvectors

Consider an operator A acting on a vector $\vec{v}$. What vectors keep their direction under the transformation $A\vec{v}$?

$\Rightarrow$ Eigenvectors

$$A\vec{v} = \lambda\vec{v}$$ } Keep the direction, allow norm to change.

λ: call the scale factor the eigenvalue

How do we find them?

$$A\vec{v} - \lambda\vec{v} = 0$$
$$A\vec{v} - \lambda \cdot I \cdot \vec{v} = 0$$
$$(A - \lambda I) \cdot \vec{v} = 0$$

$\vec{v}$, in order to be general, can't be zero.

$$|A - \lambda I| = 0$$ } determinant.

Ex) $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ $\qquad |A - \lambda I| = \left| \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right|$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = -\lambda(-3-\lambda) + 2$$

$$3\lambda + \lambda^2 + 2 = 0$$

$$\underbrace{\lambda^2 + 3\lambda + 2 = 0}_{\text{characteristic polynomial}} \qquad \lambda = -2, -1$$ } Eigenvalues

Let $\lambda_1 = -1$, $\lambda_2 = -2$.

Going back to the original equation: $A\vec{v} = \lambda_1 \vec{v}$

$$\Rightarrow \quad (A - \lambda_1) \cdot \vec{v} = 0$$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \left. \begin{array}{l} v_1 + v_2 = 0 \\ -2v_1 - 2v_2 = 0 \end{array} \right\} \text{ gives } \quad v_1 = -v_2$$

Eigenvector is then $\vec{v}_1 = a \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2^{nd} Eigenvalue: $\lambda_2 = -2$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \quad \Rightarrow \quad 2v_1 + v_2 = 0$$

$$\Rightarrow \quad 2v_1 = -v_2$$

twice as big.

so $\vec{v}_2 = b \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Eigenvectors: $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix} \right\}$

Normalized!

Change of Basis

Ex) Consider an operator A in its basis, given by basis vectors $|a_i\rangle$

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Consider a new basis $|b_i\rangle$ defined by:

$$\langle a_i | b_1 \rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \langle a_i | b_2 \rangle = \begin{pmatrix} -i/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$$

What is A in the new basis?

$$A' = U^{\dagger} A U$$, where U is the change of basis matrix.

$$U = \begin{pmatrix} \langle a_1|b_1\rangle & \langle a_1|b_2\rangle \\ \langle a_2|b_1\rangle & \langle a_2|b_2\rangle \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

$$U^{\dagger} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix}$$

$$A' = \overbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -i/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} i/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -i/\sqrt{2} \\ 1/\sqrt{2} & i/\sqrt{2} \end{pmatrix}$$

$$A' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Diagonalizing a Matrix

A matrix is diagonal in the basis of its eigenvectors.

It can only be diagonalized if you have a set of linearly independent eigenvectors.

The eigenvectors and eigenbasis characterize an operator; they are a geometric representation of the operator.

Ex) Consider diagonalizing an operator A, into an operator D.

$$A' = U^{\dagger} A U \iff D = E^{\dagger} A E$$

In Dirac notation: $D = \langle e_i | A | e_j \rangle$ $\quad |e_j\rangle$ are eigenvectors of A.

Let $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ in some basis.

$$|A - \lambda I| = 0 = \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) - 8$$

$$3 - 4\lambda + \lambda^2 - 8 = 0 = \lambda^2 - 4\lambda - 5 \qquad \lambda_1 = -5 \quad \lambda_2 = 1$$