The Postulates of Quantum Mechanics

When people say we don't understand quantum mechanics, that isn't entirely true. We understand the mathematics – it is linear algebra. What we really don't understand is *why* it works, and the *physical* interpretation.

The mathematical description of Q.M. is summarized in 5 *postulates* of how interactions work at small scales.

Postulate 1: The state of a Q.M. system is completely specified by $\Psi(\vec{x},t)$ that depends on the coordinates of the particles and on time.

$\Rightarrow$ $\Psi(\vec{x},t)$ is the wave function.

$\Rightarrow$ $\Psi^*(\vec{x},t)\,\Psi(\vec{x},t)\,d^3\vec{x}$ is the prob. that a particle is in the volume $d^3\vec{x}$ at $(\vec{x},t)$.

$\Rightarrow$ $\int_{-\infty}^{\infty} \Psi^*(\vec{x},t)\,\Psi(\vec{x},t)\,d^3\vec{x} = 1$ normalizability.

Postulate 2: Every observable in classical mechanics corresponds to a linear, Hermitian operator in Q.M.

$\Rightarrow$ Must be Hermitian in order to have *real* observables.

$\Rightarrow$ Classical: p

Q.M.: $-i\hbar \frac{\partial}{\partial x} = \hat{p}$

Postulate 3: Any measurement of the observable $\hat{O}$ results in an eigenvalue of $\hat{O}$, acting on the wavefunction Ψ.

$$\underbrace{\hat{O}}_{\text{operator}}|\Psi\rangle = \underbrace{\lambda}_{\text{observed measurements}}|\Psi\rangle$$

* This is the heart of Q.M., this is how we observe "quanta" or discreteness!! *

Note: Could start in arbitrary state, which you will learn about later.

Note: Once measurement of $|\psi\rangle$ yields λ_i, we say the wavefunction collapses into the eigenstate $|\psi_i\rangle$

Postulate 4: If a system is in a normalized state Ψ, then the average value of the observable $\hat{O}$ is given by,

$$\langle O \rangle = \int_{-\infty}^{\infty} \Psi^* \hat{O} \Psi \, d^3\vec{x}$$

Postulate 5: The wavefunction evolves in time according to the time-dependent Schrödinger equation:

$$\hat{H} \Psi(\vec{x},t) = i\hbar \frac{\partial \Psi}{\partial t}$$

or $$i\hbar \cdot \frac{\partial \Psi(\vec{x},t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x},t) + V(\vec{x}) \Psi(\vec{x},t)$$

* Quick Thought: Set $V(\vec{x}) = 0$.

$$\Rightarrow \quad i\cancel{\hbar} \frac{\partial \Psi(\vec{x},t)}{\partial t} = -\frac{\hbar^{\cancel{2}}}{2m} \nabla^2 \Psi(\vec{x},t)$$

$$\frac{\partial \Psi(\vec{x},t)}{\partial t} = \frac{i\hbar}{2m} \nabla^2 \Psi(\vec{x},t)$$ } What does this look like to you?

$$\frac{\partial f(\vec{x},t)}{\partial t} = D \nabla^2 f(\vec{x},t)$$ } Diffusion equation! Diffusivity D.

Here $D = \frac{i\hbar}{2m}$ imaginary diffusion strength!

The Schrödinger equation is a diffusion equation with <u>imaginary diffusivity</u>.

Integral Tips & Tricks

You probably know the trick that if you have the integral:

$$I = \int_{-\infty}^{\infty} f(x)\,dx$$ where $f(x)$ is an even function (symmetric function).

$$\Rightarrow \quad I = 2\int_{0}^{\infty} f(x)\,dx$$

What if $f(x)$ is an odd function? (antisymmetric)

$$I = \int_{-\infty}^{\infty} f(x)\,dx = 0$$

Why is this useful? Take a function, that could be a wavefunction:

$$f(x) = Ne^{-x^2}$$

What is the expectation value of position, $\langle x \rangle$?

$$\langle x \rangle \equiv \int_{-\infty}^{\infty} Nxe^{-x^2}\,dx = N\underbrace{\int_{-\infty}^{\infty} xe^{-x^2}\,dx}_{\text{odd}} = 0$$

Note: An even function multiplied by an odd function is odd.

even $\equiv +$
odd $\equiv -$

even $\cdot$ odd $=$ odd
$(+)\cdot(-) = (-)$

odd $\cdot$ odd $=$ even
$(-)\cdot(-) = (+)$

even $\cdot$ even $=$ even
$(+)\cdot(+) = (+)$

Important Point About Commutation

Recall, I discussed what the eigenbasis of an operator $\hat{B}$, and how to find it.

Imagine you have two observables $\hat{A}$ & $\hat{B}$, and perform the following measurements (rapidly):

$$\underset{1}{\hat{A}} \rightarrow \underset{2}{\hat{B}} \rightarrow \underset{3}{\hat{A}}$$

Will the measurement in 1 be equal to that in 3?

Classically it would, because measurement does not affect the system.

In Q.M., you will only have measurement 1 and 3 equal if A and B commute. $\rightarrow$ $[\hat{A},\hat{B}]=0$

Recall: Earlier in the postulates I said measurement collapses the wave function into an eigenstate $|\psi_i\rangle$, with measured value λ_i.

In our example, $\hat{A}$ puts the system in some eigenstate $|A_j\rangle$, then $\hat{B}$ puts it in $|B_j\rangle$.

Measurement 3 will only be A_j (eigenvalue) if

$$|A_j\rangle = |B_j\rangle \Big\}$$ These are eigenvectors! $\hat{A}$ & $\hat{B}$ have to <u>share an eigenbasis</u>.

Commutation implies a common eigenbasis $[\hat{A},\hat{B}]=0$

(The first question in your homework asks to show this.)