Why can we model most things as harmonic oscillators?

Recall, we have written the equation for simple harmonic motion as:

$$(1) \qquad \frac{d^2 x(t)}{dt^2} = -\omega^2 x(t)$$

We also related this classical equation to more quantum mechanical ideas of eigenfunctions and eigenvalues.

Identify the operator $\mathcal{L} \equiv \frac{d^2}{dt^2}$ and "eigenvalue" $\lambda \equiv -\omega^2$

(1) becomes: $\mathcal{L} x(t) = \lambda x(t)$ $\quad$ $x(t)$ are then the eigenfunctions.

* You've been doing this since 1st year physics!

(1) Came from Newton's laws, where $\Sigma F = ma$.

$$\Sigma F = ma = \underbrace{-kx}$$

Where does this come from?
Why is the force proportional to the distance?

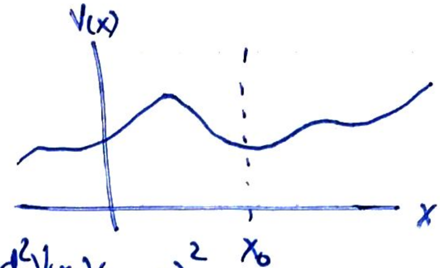
In Schrodinger's equation we deal with potentials: $V(x)$.

$$H\Psi = E\Psi \quad \} \text{ stationary states}$$

$$-\frac{\hbar}{2m}\frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x)$$

Most of physics is concerned with potential minima.

Consider an arbitrary potential $V(x)$, and a point x_0 in the neighbourhood of a potential minimum.



Taylor expand: $V(x) = V(x_0) + \frac{dV(x_0)}{dx}(x-x_0) + \frac{1}{2}\frac{d^2V(x_0)}{dx^2}(x-x_0)^2 + \ldots$

Recall only differences in potentials matter:

$$\Delta V_{(x)} = V(x) - V(x_0) = \frac{dV(x_0)}{dx}(x-x_0) + \frac{1}{2}\frac{d^2V(x_0)}{dx^2}(x-x_0)^2 + \ldots$$

Now since $V(x_0)$ is a minimum, $\frac{dV(x_0)}{dx} = 0$

$$\Rightarrow \quad \Delta V(x) = \frac{1}{2}\frac{d^2V(x_0)}{dx^2}(x-x_0)^2$$

Let $K \equiv \frac{d^2V(x_0)}{dx^2}$ } constant.

$$\Delta V \cong \frac{1}{2}K\Delta x^2$$

If there is no friction, $\vec{F} = -\vec{\nabla} V$

1D spring $\quad F = -\frac{dV}{dx} = -Kx$!

* You can see any motion localized to a neighbourhood of a potential minimum will act like simple harmonic motion! *

Separation of Variables

A powerful method for seeking solutions to partial differential equations.

In E&M you should have learned that, in the absence of sources, the electric potential satisfies Laplace's equation.

$$\nabla^2 V(x,y,z) = 0.$$

You might not encounter this exact equation in this course, but the method is the same.

Example 3.3 Griffiths E&M. 3rd Edition

Ex: Consider two infinite grounded metal plates, that are parallel to the xz plane. One is at $y=0$, other at $y=a$

The left end at $x=0$ is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. What is the potential in the slot?

Solution: Does not depend on z.

$$\nabla^2 V(x,y) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Step 1: Seek separable solutions: $V(x,y) = A(x)B(y)$

Step 2: Try them out!

$$\frac{\partial V}{\partial x} = \frac{\partial A(x)}{\partial x} B(y) \qquad \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 A(x)}{\partial x^2} B(y)$$

Similarly: $\frac{\partial^2 V}{\partial y^2} = A(x) \frac{\partial^2 B(y)}{\partial y^2}$

$$\Rightarrow \quad \frac{\partial^2 A(x)}{\partial x^2} B(y) + A(x) \frac{\partial^2 B(y)}{\partial y^2} = 0$$

Can change partials into full derivatives: $\frac{d^2A(x)}{dx^2}B(y) + A(x)\frac{d^2B(y)}{dy^2} = 0$

Step 3: separate the variables (this means make each term only depend on 1 variable)

$$\Rightarrow \underbrace{\frac{1}{A(x)}\frac{d^2A(x)}{dx^2}}_{f(x)} + \underbrace{\frac{1}{B(y)}\frac{d^2B(y)}{dy^2}}_{g(y)} = 0$$

$$f(x) + g(y) = 0$$

Step 4: Crucial subtle step.

Answer: $f(x) = C_1 \quad g(y) = C_2$ where $C_1 + C_2 = 0$, $C_1, C_2 \in \mathbb{R}$ constants.

Why?: Imagine $f(x)$ varied in the equation:

$$f(x) + g(y) = 0.$$

If $f(x)$ changed, then $g(y)$ would have to compensate to keep the equality with zero.

However, in that case, $g(y)$ would depend on x!

$\therefore$ $f(x)$ & $g(y)$ must be constant.

$$\Rightarrow \frac{1}{A(x)}\frac{d^2A(x)}{dx^2} = C_1 \qquad \frac{1}{B(y)}\frac{d^2B(y)}{dy^2} = C_2$$

Try $C_1 > 0$, $C_2 < 0$.

* $C_1 > 0$ because we don't expect stable wave solutions for an open boundary (x @ infinity).

$$\Rightarrow \quad \frac{d^2A(x)}{dx^2} = K^2 A(x) \qquad \frac{d^2B(y)}{dy^2} = -K^2 B(y)$$

Recall: $A(x) = Ce^{Kx} + De^{-Kx}$ $\qquad B(y) = N\sin(Ky) + M\cos(Ky)$

$$\Rightarrow \quad V(x,y) = A(x)B(y)$$
$$= \left(Ce^{Kx} + De^{-Kx}\right)\left(N\sin(Ky) + M\cos(Ky)\right)$$

<u>Step 5</u>: Boundary conditions.

* $\lim_{x\to\infty} V(x,y) = 0 \quad \Rightarrow \quad C = 0$

$$\Rightarrow \quad A(x) = De^{-Kx}$$

$$\Rightarrow \quad V(x,y) = e^{-Kx}\left(N\sin(Ky) + M\cos(Ky)\right)$$ absorb D into N, M.

* $V(x,0) = 0 \quad (@\ y=0) \quad \Rightarrow \quad M = 0.$

$$\Rightarrow \quad V(x,y) = Ne^{-Kx}\sin(Ky)$$

* $V(x,a) = 0 \qquad \sin(Ka) = 0$ when $Ka = \pi n$, $n = 1, 2, 3, \ldots$

$$\Rightarrow K = \frac{\pi}{a} n$$

$$\therefore \quad V(x,y) = Ne^{-\frac{\pi n}{a}x}\sin\left(\frac{\pi}{a}ny\right)$$ Stop here for now...

* This problem is more complicated than your homework, because we can't fit the boundary condition

$V(0,y) = V_0(y)$ at $x=0$. Would be easier if $V(0,y) = 0$.