Brief Points on Observables

Recall: Stern - Gerlach experiment. Showed quantization of spin.

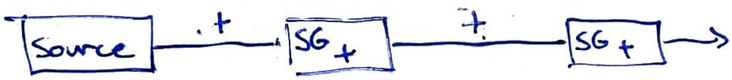
Prepare atoms in spin state $S_z = \frac{1}{2}\hbar$ ⇒ atom would always have S_z measured as $\frac{1}{2}\hbar$.

In Q.M. we choose to label this state as $|\underbrace{S_z = \tfrac{1}{2}\hbar}_{\text{label}}\rangle$ } Vector in an abstract space.

* For brevity, label as $|+\rangle$. *

Could also have $S_z = -\frac{1}{2}\hbar$, label as $|-\rangle$.

If atoms were "prepared" in $|+\rangle$, you would not expect to then measure $|-\rangle$.



etc... never see "–" in this config.

So $|+\rangle$ and $|-\rangle$ are mutually exclusive, and are the only two possibilities for the z-component of the spin.

* Important: It has to be $|+\rangle$ or $|-\rangle$ for S_z because that covers both sides of the "z-axis".

Complete, orthogonal set ⇒ $|+\rangle$ and $|-\rangle$ span the state space.

An atom in any spin state $|S\rangle$ is a linear combination of $|+\rangle, |-\rangle$.

$$|S\rangle = \langle +|S\rangle |+\rangle + \langle -|S\rangle |-\rangle$$

The properties of the "observable" $S_z = \pm \frac{1}{2}\hbar$ can be described by Hermitean operators.

eigenvalues.

$$S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle$$

operator eigenvectors

$S_z|+\rangle$ is not a measurement, it tells you how the observable operator acts on $|+\rangle$.

An atom could be in any state $|S\rangle$, with probabilities associated with the spin.

Ex: $|S\rangle = c_1|+\rangle + c_2|-\rangle$

$|c_1|^2, |c_2|^2$ are probabilities. *assume normalized!

If we measured S, we would get two possibilities:

$$\left.\begin{aligned} \langle +|S\rangle = c_1 \quad &\Rightarrow \quad |\langle +|S\rangle|^2 = c_1^2 \\ \langle -|S\rangle = c_2 \quad &\Rightarrow \quad |\langle -|S\rangle|^2 = c_2^2 \end{aligned}\right\} \text{probability of finding either.}$$

Expectation value of the measurement:

$\langle S_z\rangle = 0.$ (Show this using $S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$)

Position Eigenstates:

Consider a state $|\Psi\rangle$ and its position-space representation:

$\Psi(x) = \langle x|\Psi\rangle$ Is $|\Psi\rangle$ a position eigenstate? Does it have corresponding eigenvalue x?

$\Rightarrow$ No, absolutely not.

You can determine position eigenstates:

$x\,\phi(x) = \lambda\,\phi(x)$ What are $\phi(x)$'s?

Remember to think about what this means in English.

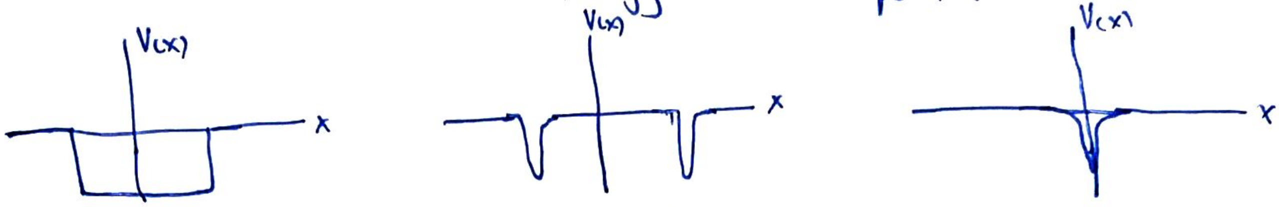
"What function $\phi(x)$ when I multiply by x gives me a number λ times the same function?"

$\Rightarrow \phi(x) = 0$? Mostly, but could be $\phi = 1$ when $\lambda = x$.

$\Rightarrow \quad \phi_\lambda(x) = A\,\delta(x-\lambda) \leftarrow$ position eigenstate is a delta function.

Solving Schrödinger's Equation

Potential $V(x)$ and Energy E are important.



square-well Double-delta Single-delta.

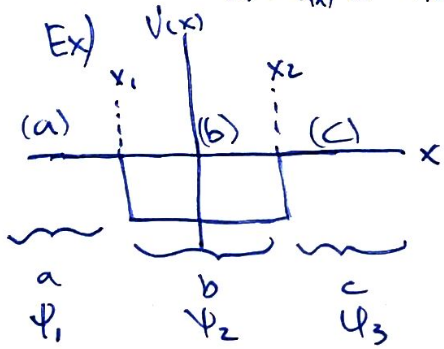
S.E. $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$

Procedure is always the same: Bound states $E<0$
Scattering states $E>0$

Solve the equation in regions of space. The equation holds for all x, we only require:

1) Continuity of $\psi(x)$
2) $\frac{d\psi}{dx}$ is continuous for all x, except when $V(x)=\infty$
3) $\psi(x)$ is normalizable (for bound states).

Ex)



Regions: $x < x_1$
$x_1 < x < x_2$
$x > x_2$

Continuity: $\psi_1(x_1) = \psi_2(x_2)$
etc.

Consider single delta function: $V(x) = -\alpha\,\delta(x)$ (*Griffith's QM Sec. 2.5)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} - \alpha\,\delta(x)\,\psi(x) = E\,\psi(x)$$

Bound states: $E<0$. Remember to break up into regions.

$x<0$: $V(x)=0$

$$\frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi = \kappa^2\psi \qquad * \; E<0 \qquad \kappa \equiv \frac{\sqrt{-2mE}}{\hbar}$$

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x} \qquad \lim_{x\to-\infty}\psi(x) = 0 \Rightarrow A=0$$

$$\psi_1(x) = Be^{\kappa x}, \quad x<0.$$

$x>0$: $V(x)=0$ Same as above, except keep first term.

$$\psi_2(x) = De^{-\kappa x}, \quad x>0.$$

We require $\psi_1(0) = \psi_2(0)$. $\Rightarrow$ $B=D$

$$\Rightarrow \psi(x) = B\begin{cases} e^{\kappa x}, & x\le 0 \\ e^{-\kappa x}, & x\ge 0 \end{cases}$$

What happens to $\frac{d\psi}{dx}$ at $x=0$? It must be discontinuous, but how discontinuous?

Integrate S.E. from $-\epsilon$ to ϵ; take limit $\epsilon\to 0$.

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}\,dx + \int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx = \underbrace{\cancel{E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx}}_{=0 \text{ as } \epsilon\to 0}$$

$$\lim_{\epsilon\to 0}\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{+\epsilon} = -\frac{2m\alpha}{\hbar^2}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\delta(x)\psi(x)\,dx = -\frac{2m\alpha\,\psi(0)}{\hbar^2}$$

$$\lim_{\epsilon\to 0}\left(\left.\frac{d\psi}{dx}\right|_{+\epsilon} - \left.\frac{d\psi}{dx}\right|_{-\epsilon}\right) = -\frac{2m\alpha}{\hbar^2}\psi(0) \qquad \psi(0) = B \qquad \lim_{\epsilon\to 0}\left.\frac{d\psi}{dx}\right|_{+\epsilon} = -B\kappa$$

$$\cancel{-2}\cancel{B}\kappa = \cancel{-}\frac{2m\alpha}{\hbar^2}\cancel{B}$$

$$\kappa = \frac{m\alpha}{\hbar^2} \Rightarrow E = -\frac{m\alpha^2}{2\hbar^2}$$

Normalize $\psi(x)$.

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 2|B|^2 \int_0^{\infty} e^{-2Kx} \, dx = \frac{|B|^2}{K} = 1$$

$$\Rightarrow \quad B = \sqrt{K} = \frac{\sqrt{m\alpha}}{\hbar}$$

good thing we know this now.

$$\Rightarrow \quad \boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-\frac{m\alpha|x|}{\hbar^2}} \; ; \quad E = -\frac{m\alpha^2}{2\hbar^2}}$$