Example 3.8 Griffith's QM

Imagine you have a system with two linearly independent states.

$$|1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{\&} \quad |2\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Suppose you have a Hamiltonian of the form:

$$H \doteq \begin{pmatrix} h & g \\ g & h \end{pmatrix} \qquad \text{where } h, g \in \mathbb{R}$$

Question: If the system starts out in state $|1\rangle$ at $t=0$, what is its state $|s\rangle$ at time t?

* It is really important to note that $|1\rangle$ and $|2\rangle$ are not necessarily eigenvectors of H.

$|1\rangle$ & $|2\rangle$ are two states that the system can be in.

The most general state is:

$$|S\rangle = a|1\rangle + b|2\rangle \doteq \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{with } |a|^2+|b|^2 = 1$$

We want to find solutions to the time-independent Schrödinger equation. $H|s\rangle = E|s\rangle$

Why?: If we can find the stationary states, it is very easy to find the time-dependent solutions!

→ Find the eigenvectors and eigenvalues of H.

$$\begin{vmatrix} h-E & g \\ g & h-E \end{vmatrix} = (h-E)^2 - g^2 = 0$$

or $\quad h-E = g \quad \Rightarrow \quad E = h-g$

but could have $\quad E = h+g$

$$\Rightarrow \quad E_\pm = h \pm g \qquad \text{These are the energies!}$$

What about the eigenvectors? $H|s_+\rangle = E_+|s_+\rangle$

$H|s_-\rangle = E_-|s_-\rangle$

$$\Rightarrow \begin{pmatrix} h & g \\ g & h \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (h \pm g)\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$h\alpha + g\beta = (h \pm g)\alpha$

$\Rightarrow \beta = \pm\alpha$

So our eigenvectors are $|s_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$

Recall that the state starts out in $|1\rangle$ at $t=0$.

$$|S(0)\rangle = |1\rangle = \underbrace{\frac{1}{\sqrt{2}}(|s_+\rangle + |s_-\rangle)}_{\text{Expressed in eigenbasis of } H.}$$

To get time-dependence, we need $e^{-\frac{iE_n t}{\hbar}}$ on each term!

$$\Rightarrow |S(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-\frac{i(h+g)t}{\hbar}}|s_+\rangle + e^{-\frac{i(h-g)t}{\hbar}}|s_-\rangle\right]$$

state at time t, if it starts in $|1\rangle$.

* What if $h=1$ and $g=0$?

$\Rightarrow |S(t)\rangle = |s_+\rangle$ because now $|s_+\rangle = |1\rangle$ and it always stays in the eigenstate!