Final Tutorial

Example 2.2 Griffith's Q.M.

Non-Stationary States:

Recall: Completeness $\quad f(x) = \sum_{n=1}^{\infty} c_n \psi_n(x)$

Fourier's Trick : $\quad c_n = \int \psi_n^*(x)\, f(x)\, dx$

Question:

Say a particle in an infinite square well potential has an initial wave function:

$$\Psi(x,0) = Ax(a-x), \qquad (0 \leq x \leq a) \qquad A \text{ is a constant.}$$

Find $\Psi(x,t)$.

Solution: Normalize! $\quad 1 = \int_0^a |\Psi(x,0)|^2 dx = |A|^2 \int_0^a x^2 (a-x)^2 dx$

$$\Rightarrow |A|^2 \frac{a^5}{30} = 1 \quad \Rightarrow \quad A = \left(\frac{30}{a^5}\right)^{1/2}$$

What do we really want to find here? We want to expand $\Psi(x,0)$ in the basis of the eigenstates of the infinite square well.

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x) \quad \Rightarrow \quad c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

If we expand successfully, time dependence is easy to introduce, through $e^{-iE_n t/\hbar}$.

$$\Rightarrow c_n = \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sqrt{\frac{30}{a^5}}\, x(a-x)\, dx$$

Integration by parts ...

$$\Rightarrow \quad c_n = \frac{4\sqrt{15}}{(n\pi)^3} \left[\cos(0) - \cos(n\pi)\right]$$

$$c_n = \begin{cases} 0 & , n \text{ even} \\ 8\sqrt{15}/(n\pi)^3 & , n \text{ odd} \end{cases} \qquad \text{Now } \Psi(x,t) = \sum_n c_n \psi_n(x)\phi_n(t)$$

$$\Rightarrow \Psi(x,t) = \underbrace{\sqrt{\frac{30}{a}}\left(\frac{2}{\pi}\right)^3 \sum_{n=1,3,5} \frac{1}{n^3}}_{c_n} \underbrace{\sin\left(\frac{n\pi}{a}x\right)}_{\psi_n(x)} \underbrace{e^{-\frac{in^2\pi^2\hbar t}{ma}}}_{\phi(t)}$$

<u>Magnetic field Sweep</u> (Assignment #5, part (b)).

You have a spin 1/2 particle in a time-dependent magnetic field.

$$\vec{B} = B\tanh(t/\tau)\,\hat{z}.$$

The Hamiltonian is $H = -\vec{\mu}\cdot\vec{B}$, $\vec{\mu} = -g\frac{e}{2mc}\vec{S}$

How do we solve time-dependent S.E. with time-dependent Hamiltonian?

$$i\hbar \frac{\partial}{\partial t}|\psi,t\rangle = H|\psi,t\rangle$$

$$H = -\vec{\mu}\cdot\vec{B} = +\frac{ge}{2mc}\vec{S}\cdot\vec{B} = \frac{ge}{2mc}B(t)\,S_z = \quad (t)\,S_z \qquad \text{where } \omega(t) \equiv \frac{geB(t)}{2mc}$$

$$H \doteq \frac{\hbar\omega(t)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Big\} \text{ In } S_z \text{ eigenstates basis.}$$

As before, any arbitrary state is given by: $|\psi,t\rangle = c_1(t)|S_z+\rangle + c_2(t)|S_z-\rangle$

$$\Rightarrow |\psi,t\rangle \doteq \begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

$$\rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \frac{\hbar\omega(t)}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix}$$

$$\Rightarrow \dot{c}_1(t) = -\frac{i\omega(t)}{2}c_1(t) \qquad \Rightarrow \qquad c_1(t) = K_1 e^{-i/2\int_0^t \omega(t')dt'}$$

$$\dot{c}_2(t) = \frac{i\omega(t)}{2}c_2(t) \qquad\qquad c_2(t) = K_2 e^{-i/2\int_0^t \omega(t')dt'}$$