Einstein Notation & Notation in General

You're most likely used to vector notation:

$$\vec{v} = (\vec{v}\cdot\hat{e}_x)\,\hat{e}_x + (\vec{v}\cdot\hat{e}_y)\,\hat{e}_y \qquad \hat{e}_i\cdot\hat{e}_j = \delta_{ij}$$

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y$$

$$\vec{a}\times\vec{b} = (a_y b_z - b_y a_z)\hat{x} - (a_x b_z - b_x a_z)\hat{y} + (a_x b_y - b_x a_y)\hat{z}$$

$$A\vec{a} = \vec{b}$$

Now you've been introduced to bra-ket notation:

$$|v\rangle = \langle e_1|v\rangle\,|e_1\rangle + \langle e_2|v\rangle\,|e_2\rangle \qquad \langle e_i|e_j\rangle = \delta_{ij}$$

order matters

$$\hat{A}|v\rangle = \lambda|v\rangle$$

$$\langle a|b\rangle = a_1 b_1 + a_2 b_2$$

cross product?? (Sparavigna & Marazzato 2016)

$$\hat{A}|a\rangle = |b\rangle$$

In this course it may be the first time you have to use the direct product:

$$\vec{a}\otimes\vec{b} = \vec{a}\otimes\vec{b}^{\,T} = \begin{bmatrix} a_1\,\vec{b}^{\,T} \\ a_2\,\vec{b}^{\,T} \end{bmatrix} = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix}$$

$$|a\rangle\langle b| = \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix} \qquad |e_1\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$e_{1,1} = 1 \qquad e_{1,2} = 0$$

$$\therefore\ |e_1\rangle\langle e_1| = ? \qquad \begin{bmatrix} e_{1,1}e_{1,1} & e_{1,1}\cancel{e_{1,2}}^{\,0} \\ e_{1,1}\cancel{e_{1,2}}^{\,0} & e_{1,2}\cancel{e_{1,2}}^{\,0} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

vector

$$\Rightarrow\ \mathbb{1} = \sum_i |e_i\rangle\langle e_i|$$

Direct products are useful for projection operators.

Assume you have an orthonormal basis $\{|e_i\rangle\}$ of a vector space V. Choose one element $|e_n\rangle$

$$\Rightarrow \quad P_n \equiv |e_n\rangle\underbrace{\langle e_n|}_{\text{just waiting to be multiplied}}$$

If you have a vector $|v\rangle \in V$, you can project it along $|e_n\rangle$:

$$P_n|v\rangle = |e_n\rangle\langle e_n|v\rangle = \underbrace{\langle e_n|v\rangle}_{\substack{\text{amount of}\\ |v\rangle \text{ in } |e_n\rangle\\ \text{direction}}} \underbrace{|e_n\rangle}_{|e_n\rangle \text{ direction.}}$$

You are in a subspace of V now. In general:

$$P_{1,\dots,N} = \sum_{n=1}^{N} |e_n\rangle\langle e_n|$$

You have seen this in your homework when you looked at spectral decomposition.

$$A = \sum_i \lambda_i |e_i\rangle\langle e_i|$$

eigenvectors

eigenvalues

Can decompose linear operator into the sum of its eigenvalues multiplied by the projection operator to its eigenspace.

Einstein Notation

A fantastic notation that only deals with components.

v_i — a vector $\vec{v}$'s component i represents the whole vector simultaneously.

$$A_i^{\ j} v_j = \lambda v_i$$

$$\underbrace{a_i b_i}_{\text{implied summation}} = \sum_i a_i b_i = a_1 b_1 + \dots + a_n b_n$$

$$\underbrace{\epsilon_{ijk} a_j b_k}_{\text{cross product}} = \sum_{j,k} \epsilon_{ijk} a_j b_k$$

ϵ_{ijk} Levi-Civita tensor.

$\epsilon_{123} = +1$

if two indices the same $\epsilon_{ijk} = 0$

swap two indices multiply by -1

e.g. $\epsilon_{213} = -1$

$$\underbrace{A_i^{\ j} a_j}_{\text{matrix multiplication}} = b_i$$

What about the direct product? Easy!

$a_i b_j \}$ components of a matrix with indices i, j

Example: How can you rewrite $\vec{a} \cdot (\vec{b} \times \vec{c})$?

$$a_i \epsilon_{ijk} b_j c_k = -b_j \epsilon_{jik} a_i c_k = -\vec{b} \cdot (\vec{a} \times \vec{c})$$

Example: Show $\vec{a} \cdot \vec{H} \cdot \vec{a} = \vec{H} \cdot (\vec{a} \otimes \vec{a})$

$$a_i H_{ij} a_j = H_{ij} a_i a_j = \vec{H} \cdot (\vec{a} \otimes \vec{a})$$

What is a vector?

Scalar: $Q' = Q$ invariant w.r.t. coordinates. Rank = 0

Vector: $V'_j = A_j^{\ i} V_i$ Rank = 1

Tensor: $T'^{ij} = A^i_{\ l} A^j_{\ m} T^{lm}$ Rank = 2

etc. : $T'^{ijk} = A^i_{\ l} A^j_{\ m} A^k_{\ n} T^{lmn}$ Rank = 3

What are the A operators?

If you have a coordinate system (q^1, q^2, q^3) and go to a new coordinate system (x^1, x^2, x^3)

$$dx^i = \underbrace{\frac{\partial x^i}{\partial q^j}} dq^j \Big\} \text{ implied summation. chain rule.}$$

$$dx^i = A^i_{\ j} dq^j \qquad V'_j = \frac{\partial q^j}{\partial x^i} V_i \Big\} \text{ covariant}$$

And tensors: $$T'^{ij} = \underbrace{\underbrace{\frac{\partial x^i}{\partial q^l}}_{\text{Jacobian}} \frac{\partial x^j}{\partial q^m}}_{\text{direct Product}} T^{lm} \Big\} \text{ contravariant}$$

$$V'^j = \frac{\partial x^j}{\partial q^i} V^i \Big\} \text{ contravariant (against)} \Rightarrow \text{dual vector}$$

ex: $\vec{E} = \frac{\partial \phi}{\partial x^j}$ ← doesn't scale with = contravariant

$$V'_j = \frac{\partial q^i}{\partial x^j} V_i \Big\} \text{ covariant (with)}$$

ex: $\vec{V} = \frac{d\vec{x}}{dt}$ ← covariant