Delta Function Continuity

Whenever you come across a delta function potential you need to make sure the $d\psi/dx$ is continuous.

Integrate the S.E. across a tiny region $\Delta\epsilon$.

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon}\frac{d^2\psi}{dx^2}\,dx + \int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx = E\int_{-\epsilon}^{+\epsilon}\psi(x)\,dx$$

$\psi(x)$ is continuous $\therefore \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon}\psi(x)\,dx = 0$

Will take the limit as $\epsilon \to 0$.

$$\Rightarrow -\frac{\hbar^2}{2m}\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} + \int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx = 0$$

$$\Rightarrow \left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} = \frac{2m}{\hbar^2}\int_{-\epsilon}^{+\epsilon}V(x)\psi(x)\,dx$$

If $V(x) = -\alpha\delta(x)$ things get simpler:

$$\lim_{\epsilon\to 0}\left.\frac{d\psi}{dx}\right|_{-\epsilon}^{\epsilon} = -\frac{2m\alpha}{\hbar^2}\psi(0)$$

Harmonic Oscillator Operators

In the harmonic oscillator: $X = \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$

$$P = i\sqrt{\frac{\hbar\omega m}{2}}\,(a_+ - a_+)$$

$* \quad H\psi = E\psi$

$$\frac{1}{2m}\left[p^2 + (m\omega x)^2\right]\psi = E\psi$$

$$\Rightarrow a_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp ip + m\omega x)$$

Definition: $a_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a_-|n\rangle = \sqrt{n}\,|n-1\rangle$

What are the matrix elements X_{nm}?

Starting with X_{nn}:

$$X_{nn} = \langle n | X | n \rangle = \langle n | \left(\sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-) \right) | n \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle n | \left(a_+ | n \rangle + a_- | n \rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle n | \left(\sqrt{n+1}\, | n+1 \rangle + \sqrt{n}\, | n-1 \rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1}\, \cancelto{0}{\langle n | n+1 \rangle} + \sqrt{n}\, \cancelto{0}{\langle n | n-1 \rangle} \right)$$

$X_{nn} = 0 \Rightarrow \langle x \rangle = 0$ as well (they are the same)

What about X_{nm}?

$$X_{nm} = \langle n | X | m \rangle = \langle n | \left(\sqrt{\frac{\hbar}{2m\omega}} (a_+ + a_-) \right) | m \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle n | \left(a_+ | m \rangle + a_- | m \rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle n | \left(\sqrt{m+1}\, | m+1 \rangle + \sqrt{m}\, | m-1 \rangle \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{m+1}\, \langle n | m+1 \rangle + \sqrt{m}\, \langle n | m-1 \rangle \right]$$

$$\boxed{X_{nm} = \sqrt{\frac{\hbar}{2m\omega}} \left[\sqrt{m+1}\, \delta_{n,m+1} + \sqrt{m}\, \delta_{n,m-1} \right]}$$

What about X^2_{nn} ?

$$X^2_{nn} = \langle n | x^2 | n \rangle = \langle n | x \cdot x | n \rangle \qquad \text{insert } \sum_m |m\rangle\langle m| = 1$$

$$= \sum_m \langle n | x | m \rangle \langle m | x | n \rangle$$

$$= \frac{\hbar}{2m\omega} \sum_m \left[\left(\sqrt{m+1}\, \delta_{n,m+1} + \sqrt{m}\, \delta_{n,m-1} \right) \left(\sqrt{n+1}\, \delta_{m,n+1} + \sqrt{n}\, \delta_{m,n-1} \right) \right]$$

$$\boxed{X^2_{nn} = \frac{\hbar}{2m\omega} (2n+1)}$$

What about X^2_{nm} ? Show $X^2_{nm} = 0$