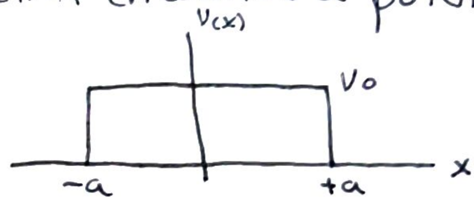
Finite Square Barrier (Griffith's Problem 2.33)

A particle travelling in the $+x$ direction encounters a potential

$$V(x) = \begin{cases} 0 & x < -a \\ V_0 & -a \leq x \leq a \\ 0 & x > a \end{cases}$$



What are the transmission coefficients?

First, consider what happens outside of the potential barrier.

$-a < x > a$: Always start with Schrodinger's Equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + \cancel{V(x)\psi(x)}^{\,0} = E\psi(x) \qquad \text{outside barrier } V(x) = 0$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2}\psi(x) \qquad \text{let } k^2 \equiv \frac{2mE}{\hbar^2}$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$$ } This differential equation governs wave function evolution outside barrier.

$x < -a$: Incoming wave + reflected wave

$$\Rightarrow \psi(x) = \underset{\substack{\uparrow \\ \text{incoming}}}{A}e^{ikx} + \underset{\substack{\uparrow \\ \text{reflected}}}{B}e^{-ikx}$$

$x > a$: Outgoing wave

$$\psi(x) = Fe^{ikx}$$

What about inside of the barrier?

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2\psi(x)}{dx^2} = -\frac{2m(V_0 - E)}{\hbar^2}\psi(x) \qquad \text{let } \Gamma^2 \equiv \frac{2m(V_0 - E)}{\hbar^2}$$

Energetics are important here, because the solution to the ODE depend on Γ (the wave period effectively).

$$\Gamma = \frac{\sqrt{2m(V_o - E)}}{\hbar}$$

What happens if $E > V_o$? Complex k.
$E < V_o$ is fine
$E = V_o$ is trivial

① For $E < V_o$: $\psi(x) = Ce^{\Gamma x} + De^{-\Gamma x}$

Now, there are 4 boundary conditions since $\psi(x)$ & $\frac{d\psi(x)}{dx}$ must be continuous everywhere.

@ $\underline{x=-a}$:

$$Ae^{-ika} + Be^{ika} = Ce^{-\Gamma a} + De^{\Gamma a} \quad (i)$$

$$ik(Ae^{-ika} - Be^{ika}) = \Gamma(Ce^{-\Gamma a} - De^{\Gamma a}) \quad (ii)$$

@ $\underline{x=a}$:

$$Ce^{\Gamma a} + De^{-\Gamma a} = Fe^{ika} \quad (iii)$$

$$\Gamma(Ce^{\Gamma a} - De^{-\Gamma a}) = ikFe^{ika} \quad (iv)$$

Transmission coefficient is $T = \frac{|F|^2}{|A|^2}$ } Only need F in terms of A

Show: $$F = \frac{2e^{-2ika}\, k\Gamma i}{2ik\Gamma\cosh(2\Gamma a) + (k^2 - \Gamma^2)\sinh(2\Gamma a)} A$$

$$\Rightarrow T = \frac{1}{\left(\frac{(\Gamma^2 + k^2)^2 \sinh^2(2\Gamma a)}{4\Gamma^2 k^2 + 1} + 1\right)}$$

$$\Rightarrow T^{-1} = 1 + \frac{V_o^2}{4E(V_o - E)} \sinh^2\left(\frac{2a}{\hbar}\sqrt{2m(V_o - E)}\right)$$

What about $E = V_0$?

$$\frac{d^2\psi}{dx^2} = 0 \quad \Rightarrow \quad \psi(x) = Cx + D \quad \text{in barrier.}$$

@ $\underline{x=-a}$: $\quad Ae^{-ika} + Be^{ika} = -Ca + D$

$$ik(Ae^{-ika} - Be^{ika}) = C$$

@ $\underline{x=a}$: $\quad Ca + D = Fe^{ika}$

$$C = ikFe^{ika}$$

Show $\quad F = \frac{e^{-2ika}}{1-ika} A$

$$\Rightarrow \quad T = \frac{1}{\left(1 + \frac{2mEa^2}{\hbar^2}\right)} \quad \Rightarrow \quad T^{-1} = 1 + \frac{2mEa^2}{\hbar^2}$$

What about $E > V_0$? $\quad \Gamma$ is complex. $\quad \Gamma = i\lambda$

$$\Rightarrow \quad \psi(x) = Ce^{i\lambda x} + De^{-i\lambda x}$$

@ $\underline{x=-a}$: $\quad Ae^{-ika} + Be^{ika} = Ce^{-i\lambda a} + De^{i\lambda a}$

$$ik(Ae^{-ika} - Be^{ika}) = i\lambda(Ce^{-i\lambda a} - De^{i\lambda a})$$

@ $\underline{x=a}$: $\quad Ce^{i\lambda a} + De^{-i\lambda a} = Fe^{ika}$

$$i\lambda(Ce^{i\lambda a} - De^{-i\lambda a}) = ikFe^{ika}$$

Show $\quad F = \frac{2k\lambda e^{-2ika}}{-i(k^2+\lambda^2)\sin(2\lambda a) + 2\lambda k\cos(2\lambda a)} A$

$$\Rightarrow \quad T = \frac{1}{\left(1 + \frac{(\lambda^2-k^2)^2\sin^2(2\lambda a)}{4\lambda^2 k^2}\right)} \quad \Rightarrow \quad T^{-1} = 1 + \frac{(\lambda^2-k^2)^2\sin^2(2\lambda a)}{4\lambda^2 k^2}$$