Spin at an Arbitrary Direction (Griffith's QM question 4.30)

Q: Construct the matrix S_r representing the component of spin angular momentum along an arbitrary direction $\hat{r}$. Use spherical coordinates:

$$\hat{r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

Find the eigenvalues and normalized eigenspinors of S_r.

A: S_r is the projection of S along $\hat{r}$.

$$S_r = S \cdot \hat{r}$$

$$= \sin\theta\cos\phi\, S_x + \sin\theta\sin\phi\, S_y + \cos\theta\, S_z$$

$$= \frac{\hbar}{2}\left[\sin\theta\cos\phi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\phi\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix}$$

Recall:

$$e^{\pm i\phi} = \cos\phi \pm i\sin\phi$$

$$S_r = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

What are the eigenvalues of S_r?

$$\begin{vmatrix} \cos\theta - \lambda & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - \lambda \end{vmatrix} = 0 = -(\cos\theta - \lambda)(\cos\theta + \lambda) - \sin^2\theta$$

$$= -(\cos^2\theta + \cancel{\lambda\cos\theta} - \cancel{\lambda\cos\theta} - \lambda^2) - \sin^2\theta$$

$$= -(\cos^2\theta + \sin^2\theta) + \lambda^2$$

$$0 = \lambda^2 - 1 \implies \lambda = \pm 1$$

but really $\lambda = \pm\frac{\hbar}{2}$

For the eigenspinors:

$$\frac{\cancel{\hbar}}{2}\begin{pmatrix} \cos\theta \pm 1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \pm 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$(\cos\theta \pm 1)a + (\sin\theta e^{-i\phi})b = 0$$

$$(\sin\theta e^{i\phi})a + (-\cos\theta \pm 1)b = 0$$

$\underline{\underline{-}}$:

$$\sin\theta e^{-i\phi} b_- = -(\cos\theta + 1)a_-$$

$$b_- = -e^{i\phi}\frac{(\cos\theta + 1)}{\sin\theta}a_-$$

$\underline{\underline{+}}$:

$$\sin\theta e^{-i\phi} b_+ = -(\cos\theta - 1)a_+$$

$$b_+ = e^{i\phi}\frac{(1-\cos\theta)}{\sin\theta}a_+$$

Use identities $\sin\theta = 2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})$

$1+\cos\theta = 2\cos^2(\theta/2)$ $\quad$ $1-\cos\theta = 2\sin^2(\theta/2)$

$$\therefore \; b_- = -e^{i\phi}\frac{\cancel{2}\cos^{\cancel{2}}(\theta/2)}{\cancel{2}\sin(\theta/2)\cancel{\cos(\theta/2)}}a_- = -e^{i\phi}\frac{\cos(\theta/2)}{\sin\theta/2}a_-$$

Choose $a_- = \sin\theta/2$

$$\Rightarrow \quad |\chi_-^{(r)}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -e^{i\phi}\cos(\theta/2) \end{pmatrix}$$

Is this normalized? Yes!

$$b_+ = e^{i\phi}\frac{\cancel{2}\sin^{\cancel{2}}(\theta/2)}{\cancel{2}\cancel{\sin(\theta/2)}\cos(\theta/2)}a_+ = e^{i\phi}\frac{\sin(\theta/2)}{\cos(\theta/2)}a_+$$

Choose $a_+ = \cos(\theta/2)$

$$\Rightarrow |\chi_+^{(r)}\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

Is this normalized? Yes!

What is $\langle\chi_+^{(r)}|\chi_-^{(r)}\rangle = ?$ $\quad \Rightarrow \quad \langle\chi_+^{(r)}|\chi_-^{(r)}\rangle = 0$

Follow-up $\underline{Q}$: Suppose a particle is in the state $|\chi_+^{(r)}\rangle$ or $|\chi_-^{(r)}\rangle$. What is the expectation value of measuring S_x? S_y? S_z?

$\underline{A}$: $\langle \chi_+^{(r)} | S_x | \chi_+^{(r)} \rangle = \frac{\hbar}{2} \left(\cos(\theta/2) \quad e^{-i\phi} \sin(\theta/2) \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ e^{i\phi} \sin(\theta/2) \end{pmatrix}$

$$= \frac{\hbar}{2} \left(\cos\theta/2 \quad e^{-i\phi} \sin\theta/2 \right) \begin{pmatrix} e^{i\phi} (\sin\theta/2) \\ \cos(\theta/2) \end{pmatrix}$$

$$= \frac{\hbar}{2} \left[e^{i\phi} \sin\theta/2 \cos\theta/2 + e^{-i\phi} \sin\theta/2 \cos\theta/2 \right]$$

$$= \frac{\hbar}{2} \sin\theta/2 \cos\theta/2 \left[e^{i\phi} + e^{-i\phi} \right]$$

$$= \hbar \sin\theta/2 \cos\theta/2 \cos\phi$$

$$\langle \chi_+^{(r)} | S_x | \chi_+^{(r)} \rangle = \frac{\hbar}{2} \sin\theta \cos\phi$$

$$\langle \chi_-^{(r)} | S_x | \chi_-^{(r)} \rangle = \frac{\hbar}{2} \left(\sin\theta/2 \quad -e^{-i\phi} \cos\theta/2 \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\theta/2 \\ -e^{i\phi} \cos\theta/2 \end{pmatrix}$$

$$= \frac{\hbar}{2} \left(\sin\theta/2 \quad -e^{-i\phi} \cos\theta/2 \right) \begin{pmatrix} -e^{i\phi} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}$$

$$= \frac{\hbar}{2} \left[-e^{i\phi} \sin\theta/2 \cos\theta/2 - e^{-i\phi} \sin\theta/2 \cos\theta/2 \right]$$

$$= -\frac{\hbar}{2} \sin\theta/2 \cos\theta/2 \left[e^{i\phi} + e^{-i\phi} \right]$$

$$= -\hbar \sin\theta/2 \cos\theta/2 \cos\phi$$

$$\langle \chi_-^{(r)} | S_x | \chi_-^{(r)} \rangle = -\frac{\hbar}{2} \sin\theta \cos\phi$$