Uncertainty, Spin.

Griffiths pr. 3.13

Q1: Prove that $[AB, C] = A[B,C] + [A,C]B$

Q2: Show that $[x^n, p] = i\hbar x^{n-1}$

Q3: Show more generally that $[f(x), p] = i\hbar \frac{df}{dx}$

A1: $[AB, C] = ABC - CAB$

$$A[B,C] + [A,C]B = A(BC - CB) + (AC - CA)B$$
$$= ABC - \cancel{ACB} + \cancel{ACB} - CAB$$
$$= ABC - CAB$$

A2: Use a test function f; $p = -i\hbar \frac{\partial}{\partial x}$

$$[x^n, p]f = -x^n i\hbar \frac{\partial f}{\partial x} + i\hbar \frac{\partial}{\partial x}(x^n f)$$
$$= -x^n i\hbar \frac{\partial f}{\partial x} + i\hbar \left(n x^{n-1} f + x^n \frac{\partial f}{\partial x}\right)$$
$$= \cancel{-x^n i\hbar \frac{\partial f}{\partial x}} + i\hbar n x^{n-1} f + \cancel{i\hbar x^n \frac{\partial f}{\partial x}}$$
$$[x^n, p]f = i\hbar n x^{n-1} f \quad \Rightarrow \quad [x^n, p] = i\hbar n x^{n-1}$$

A3: Use a test function g.

$$[f(x), p]g = -f(x) i\hbar \frac{\partial g}{\partial x} + i\hbar \frac{\partial}{\partial x}(f(x) g)$$
$$= \cancel{-i\hbar f(x) \frac{\partial g}{\partial x}} + i\hbar \frac{\partial f}{\partial x} g + \cancel{i\hbar f(x) \frac{\partial g}{\partial x}}$$
$$[f(x), p]g = i\hbar \frac{\partial f}{\partial x} g \quad \Rightarrow \quad [f(x), p] = i\hbar \frac{\partial f(x)}{\partial x}$$

Griffith's prob. 3.15

Q4: Show that two non-commuting operators cannot have a complete set of common eigenfunctions.

Hint: Show that if $\hat{P}$ and $\hat{Q}$ have a complete set of common eigenfunctions, then $[\hat{P},\hat{Q}]f = 0$ for any function in Hilbert space.

A4: Eigenfunctions of Hermitian operators form complete sets.

Any function f (in Hilbert space) can be written as a linear combination of these eigen functions.

$$\Rightarrow \qquad f = \sum_n \underset{\text{coefficients}}{c_n} \underset{\text{eigen functions}}{f_n}$$

If two operators $\hat{P}$ & $\hat{Q}$ share the same eigenspace then the f_n's are the same.

$$\therefore \quad \hat{P} f_n = p_n f_n \quad , \quad \hat{Q} f_n = q_n f_n$$

$$\hat{P}\hat{Q} f = \hat{P}\hat{Q} \sum_n c_n f_n = \hat{P}(\hat{Q} \sum_n c_n f_n) = \hat{P} \sum_n c_n q_n f_n$$

$$= \sum_n c_n q_n p_n f_n$$

$$\hat{Q}\hat{P} f = \hat{Q}\hat{P} \sum_n c_n f_n = \hat{Q}(\hat{P} \sum_n c_n f_n) = \hat{Q} \sum_n c_n p_n f_n$$

$$= \sum_n c_n p_n q_n f_n$$

$$\therefore \quad (\hat{P}\hat{Q} - \hat{Q}\hat{P}) f = 0 \quad \Rightarrow \quad [\hat{P},\hat{Q}] = 0$$

if $\hat{P},\hat{Q}$ share the same eigenbasis.

Spin: Recall $S^2 |s m\rangle = \hbar^2 s(s+1) |s m\rangle$
$S_z |s m\rangle = \hbar m |s m\rangle$

$$S_\pm |s m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\, |s\, (m\pm 1)\rangle$$

where $S_\pm = S_x \pm i S_y$

$s = 0, 1/2, 1, 3/2, \ldots \qquad m = -s, -s+1, \ldots, s-1, s$

Q5: Find the matrix representing S_x for a particle of spin 3/2 (in the S_z basis). Solve the characteristic equation to find the eigenvalues of S_x.

A5: 4×4 matrix representation $m = -3/2, -1/2, +1/2, +3/2$

$$\left.\begin{aligned} S_+ |3/2\ 3/2\rangle &= 0 \\ S_+ |\tfrac{3}{2}\ \tfrac{1}{2}\rangle &= \sqrt{3}\hbar\, |\tfrac{3}{2}\ \tfrac{3}{2}\rangle \\ S_+ |\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle &= 2\hbar\, |\tfrac{3}{2}\ \tfrac{1}{2}\rangle \\ S_+ |\tfrac{3}{2}\ {-\tfrac{3}{2}}\rangle &= \sqrt{3}\hbar\, |\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle \end{aligned}\right\} \quad S_+ = \hbar\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} S_- |\tfrac{3}{2}\ \tfrac{3}{2}\rangle &= \sqrt{3}\hbar\, |\tfrac{3}{2}\ \tfrac{1}{2}\rangle \\ S_- |\tfrac{3}{2}\ \tfrac{1}{2}\rangle &= 2\hbar\, |\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle \\ S_- |\tfrac{3}{2}\ {-\tfrac{1}{2}}\rangle &= \sqrt{3}\hbar\, |\tfrac{3}{2}\ {-\tfrac{3}{2}}\rangle \\ S_- |\tfrac{3}{2}\ {-\tfrac{3}{2}}\rangle &= 0 \end{aligned}\right\} \quad S_- = \hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

and $$S_x = \left(\frac{S_+ + S_-}{2}\right) = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}$$

$\lambda^4 - 10\lambda^2 + 9 = 0 \implies \lambda = \pm 3, \pm 1$

$\implies \pm\frac{3}{2}\hbar, \pm\frac{1}{2}\hbar$ for the eigenvalues.